Note on Evaluation of AWG Port Utilization Probabilities $d_l(1, j),\ j \neq 1, D, D-1$, and $d_l(1, 1)$

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EVALUATION OF $d_l(1, j)$ FOR $j \neq 1, D, D-1$

In this appendix we evaluate $d_l(1, j)$ for $j \neq 1, D, D-1$, i.e., the output port j is not a direct neighbor of sender port D, and note that the $d_l(1, j)$ are the same for these AWG ports j. We evaluate the probability $d_l(1, j)$ for the event that the considered multicast packet with l destinations is transmitted over the star subnetwork and if it were transmitted over the AWG it would require one packet copy transmission to the port j counting from the sender port. In particular, we evaluate the probabilities $\Delta_i,\ i = 1, \ldots, 5$, for the following five mutually exclusive events: Δ_1 is the probability for the event of interest (namely that the multicast packet is transmitted over the star subnetwork and if it were transmitted over the AWG it would require one packet copy transmission to the port j counting from the sender port) with the additional condition that at (or directly next to) the sending port there is *no* occupied RS node and *one* occupied RS segment. Δ_2 is defined analogously for *no* occupied RS node and *two* adjacent occupied RS segments. Δ_3 is defined analogously for *one* occupied RS node and *no* occupied RS segment, Δ_4 for *one* occupied RS node and *one* occupied RS segment, and Δ_5 for *one* occupied RS node and *two* occupied RS segments. We obtain

$$\begin{aligned}\Delta_1 \quad = \quad & D \sum_{s=0}^{S} \frac{\binom{S}{s}}{\binom{DS}{s}} \cdot \left\{ \sum_{k=1}^{S} (S - 1 + \frac{1}{2} + \frac{1}{2}) \frac{\binom{S-1}{k-1}}{\binom{DS}{k}} \kappa_l(s, k) \frac{1}{k} \left(1 - \frac{s}{l+1}\right) \right. \\ & + \sum_{k=2}^{S+1} (S - 1 + \frac{1}{2} + \frac{1}{2}) \frac{\binom{S-1}{k-2}}{\binom{DS}{k}} \kappa_l(s, k) \frac{1}{k} \left(1 - \frac{s}{l+1}\right) 2 \cdot \frac{1}{2} \\ & \left. + \sum_{k=3}^{S+2} (S - 1 + \frac{1}{2} + \frac{1}{2}) \frac{\binom{S-1}{k-3}}{\binom{DS}{k}} \kappa_l(s, k) \frac{1}{k} \left(1 - \frac{s}{l+1}\right) \frac{1}{4} \right\}. \end{aligned} \tag{1}$$

In (1) the factor $\binom{S}{s}/\binom{DS}{s}$ is the probability of having s RS nodes at the destination port occupied and no occupied RS nodes at any other AWG port. The factor $\binom{S-1}{k-1}/\binom{DS}{k}$ accounts for the probability of having $k-1$ of the internal segments at the destination port occupied. The factor $(S - 1 + 1/2 + 1/2)$ accounts for number of possible positions for the one occupied segment at the source port, which has

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$S-1$ internal segments, plus the two border segments, each of which is associated with the considered source port with probability 1/2. The factor $\frac{1}{k}\left(1-\frac{s}{l+1}\right)$ is the probability that the sender lies in the one occupied segment at the source port. To see this, note that the source node would fall on one of the occupied RS nodes with probability $s/(l+1)$. Hence, with probability $1-s/(l+1)$ the source node falls on one of the occupied segments. The source node falls on a particular of the k segments, namely the one occupied segment associated with the sending port with probability $[1-s/(l+1)]/k$. Similar to (**??**) the three summands in the braces account for the scenarios where the ring-homed destination nodes are (i) on the internal RS segments of the destination port, (ii) on the internal and one border segment of the destination port, and (iii) on the internal and two border segments of the destination port.

Furthermore, we have

$$\begin{aligned}\Delta_2 &= D\sum_{s=0}^{S}\frac{\binom{S}{s}}{\binom{DS}{s}}\cdot\left\{\sum_{k=2}^{S+1}S\frac{1}{4}\frac{\binom{S-1}{k-2}}{\binom{DS}{k}}\kappa_l(s,k)\frac{2}{k}\left(1-\frac{s}{l+1}\right)+\sum_{k=3}^{S+2}S\frac{1}{4}\frac{\binom{S-1}{k-3}}{\binom{DS}{k}}\kappa_l(s,k)\frac{2}{k}\left(1-\frac{s}{l+1}\right)2\cdot\frac{1}{2}\right.\\ &\qquad\left.+\sum_{k=4}^{S+3}S\frac{1}{4}\frac{\binom{S-1}{k-4}}{\binom{DS}{k}}\kappa_l(s,k)\frac{2}{k}\left(1-\frac{s}{l+1}\right)\frac{1}{4}\right\}, \qquad (2)\end{aligned}$$

The reasoning leading to (2) is analogous to the reasoning that resulted in (1) with the main difference that there are S possible ways of having two adjacent occupied segments at the source port, and with probability 1/4 these are assigned to the RS node between them. Also, the source node can now be on of the two occupied segments at the source port, which is accounted for by the factor $2/k$.

We also have

$$\Delta_3 = D\cdot\sum_{s=1}^{S+1}\frac{\binom{S}{s-1}\binom{S}{1}}{\binom{DS}{s}}\frac{1}{l+1}\left\{\sum_{k=0}^{S-1}\frac{\binom{S-1}{k}}{\binom{DS}{k}}\kappa_l(s,k)+\sum_{k=1}^{S}\frac{\binom{S-1}{k-1}}{\binom{DS}{k}}\kappa_l(s,k)+\sum_{k=2}^{S+1}\frac{\binom{S-1}{k-2}}{\binom{DS}{k}}\frac{1}{4}\kappa_l(s,k)\right\}, \qquad (3)$$

where the factor $\binom{S}{s-1}\binom{S}{1}/\binom{DS}{s}$ is the probability that there is one occupied RS (sender) node at the sending port and $s-1$ occupied RS (destination) nodes at the considered destination port. Hereby, $1/(l+1)$ is the probability that the occupied RS node at the sender port is the source node of the multicast. The reasoning for the three summands inside the braces is identical to the reasoning leading to (**??**).

$$\begin{aligned}\Delta_4 &= D\cdot\sum_{s=1}^{S+1}\frac{\binom{S}{s-1}\binom{S}{1}}{\binom{DS}{s}}\left\{\sum_{k=1}^{S}2\frac{1}{2}\frac{\binom{S-1}{k-1}}{\binom{DS}{k}}\kappa_l(s,k)\left(\frac{1}{l+1}+\frac{1}{k}\left(1-\frac{s}{l+1}\right)\right)\right.\\ &\qquad+\sum_{k=2}^{S+1}2\frac{1}{2}\frac{\binom{S-1}{k-2}}{\binom{DS}{k}}\kappa_l(s,k)\left(\frac{1}{l+1}+\frac{1}{k}\left(1-\frac{s}{l+1}\right)\right)2\frac{1}{2}\\ &\qquad\left.+\sum_{k=3}^{S+2}2\frac{1}{2}\frac{\binom{S-1}{k-3}}{\binom{DS}{k}}\kappa_l(s,k)\left(\frac{1}{l+1}+\frac{1}{k}\left(1-\frac{s}{l+1}\right)\right)\frac{1}{4}\right\}, \qquad (4)\end{aligned}$$

where there are two possible positions for placing the occupied segment adjacent to the occupied RS node, and the occupied segment is associated with the occupied RS node with probability 1/2. The factor $\left(\frac{1}{l+1}+\frac{1}{k}\left(1-\frac{s}{l+1}\right)\right)$ is the probability that the sender is either the occupied RS node or on the one occupied segment at the sender port.

Finally,

$$\Delta_5 = D \cdot \sum_{s=1}^{S+1} \frac{\binom{S}{s-1}\binom{S}{1}}{\binom{DS}{s}} \left\{ \sum_{k=2}^{S+1} \frac{1}{4} \frac{\binom{S-1}{k-2}}{\binom{DS}{k}} \kappa_l(s,k) \left(\frac{1}{l+1} + \frac{2}{k}\left(1 - \frac{s}{l+1}\right)\right) + \sum_{k=3}^{S+2} \frac{1}{4} \frac{\binom{S-1}{k-3}}{\binom{DS}{k}} \kappa_l(s,k) \left(\frac{1}{l+1} + \frac{2}{k}\left(1 - \frac{s}{l+1}\right)\right) 2\frac{1}{2} + \sum_{k=4}^{S+3} \frac{1}{4} \frac{\binom{S-1}{k-4}}{\binom{DS}{k}} \kappa_l(s,k) \left(\frac{1}{l+1} + \frac{2}{k}\left(1 - \frac{s}{l+1}\right)\right) \frac{1}{4} \right\}, \quad (5)$$

with which we obtain

$$d_l(1,j) = \sum_{i=1}^{5} \Delta_i. \quad (6)$$

EVALUATION OF $d_l(1,1)$

In this appendix we evaluate $d_l(1,1)$, which is equal to $d_l(1,D-1)$. Analogous to the evaluation of $d_l(1,j)$ above, we evaluate $d_l(1,1)$ as the sum of the probabilities for five mutually exclusive events detailed above. The main difference from the $d_l(1,j),\ j \neq 1, D, D-1$ is that now the border segment between the source port and the destination port may be assigned to the destination port and thus not be available for the positioning of the occupied segments at the source port. In addition, we need to consider a distinction between the cases $D = 2$ and $D \geq 3$. We incorporate this distinction through a factor ν which we set $\nu = 2$ for $D = 2$ and $\nu = 1$ for $D \geq 3$. We then obtain

$$\Delta_1 = D \sum_{s=0}^{S} \frac{\binom{S}{s}}{\binom{DS}{s}} \cdot \left\{ \sum_{k=1}^{S} S \frac{\binom{S-1}{k-1}}{\binom{DS}{k}} \kappa_l(s,k) \frac{1}{k}\left(1 - \frac{s}{l+1}\right) + \sum_{k=2}^{S+1} \left(S - \frac{\nu}{4}\right) \frac{\binom{S-1}{k-2}}{\binom{DS}{k}} \kappa_l(s,k) \frac{1}{k}\left(1 - \frac{s}{l+1}\right) + \sum_{k=3}^{S+2} \left(S - \frac{\nu}{2}\right) \frac{\binom{S-1}{k-3}}{\binom{DS}{k}} \kappa_l(s,k) \frac{1}{k}\left(1 - \frac{s}{l+1}\right) \frac{1}{4} \right\}. \quad (7)$$

In the scenario accounted for by the first summand in braces in (7), the $k-1$ occupied segments at the destination port are all internal segments, thus $S - 1 + 1/2 + 1/2$ positions are available for the occupied segment at the source port, as in (1). In the scenario where one of the border segments of the destination port is occupied (second summand in (7)) then there are only $S - 1 + 1/2 + 1/4$ available positions for the occupied segment at the source port, since with probability $1/2 \cdot 1/2$ the border segment is assigned to the source port and not occupied by the destination port. For the scenario where both border segments are occupied by the destination port (third summand in (7)), only $S - 1 + 1/2$ positions are available for the occupied segments at the source port.

Analogously we obtain

$$\Delta_2 = D \sum_{s=0}^{S} \frac{\binom{S}{s}}{\binom{DS}{s}} \cdot \left\{ \sum_{k=2}^{S+1} S \frac{1}{4} \frac{\binom{S-1}{k-2}}{\binom{DS}{k}} \kappa_l(s,k) \frac{2}{k}\left(1 - \frac{s}{l+1}\right) + \sum_{k=3}^{S+2} \left(\frac{S}{4} - \frac{\nu}{8}\right) \frac{\binom{S-1}{k-3}}{\binom{DS}{k}} \kappa_l(s,k) \frac{2}{k}\left(1 - \frac{s}{l+1}\right) + \sum_{k=4}^{S+3} \frac{S-\nu}{4} \frac{\binom{S-1}{k-4}}{\binom{DS}{k}} \kappa_l(s,k) \frac{2}{k}\left(1 - \frac{s}{l+1}\right) \frac{1}{4} \right\}, \quad (8)$$

$$\Delta_4 = D \cdot \sum_{s=1}^{S+1} \frac{\binom{S}{s-1}\binom{S}{1}}{\binom{DS}{s}} \left\{ \sum_{k=1}^{S} \frac{\binom{S-1}{k-1}}{\binom{DS}{k}} \kappa_l(s,k) \left(\frac{1}{l+1} + \frac{1}{k}\left(1 - \frac{s}{l+1}\right)\right) + \sum_{k=2}^{S+1} \left(1 - \frac{\nu}{4S}\right) \frac{\binom{S-1}{k-2}}{\binom{DS}{k}} \kappa_l(s,k) \left(\frac{1}{l+1} + \frac{1}{k}\left(1 - \frac{s}{l+1}\right)\right) + \sum_{k=3}^{S+2} \left(1 - \frac{\nu}{2S}\right) \frac{\binom{S-1}{k-3}}{\binom{DS}{k}} \kappa_l(s,k) \left(\frac{1}{l+1} + \frac{1}{k}\left(1 - \frac{s}{l+1}\right)\right) \frac{1}{4} \right\}, \quad (9)$$

and

$$\Delta_5 = D \cdot \sum_{s=1}^{S+1} \frac{\binom{S}{s-1}\binom{S}{1}}{\binom{DS}{s}} \left\{ \sum_{k=2}^{S+1} \frac{1}{4} \frac{\binom{S-1}{k-2}}{\binom{DS}{k}} \kappa_l(s,k) \left(\frac{1}{l+1} + \frac{2}{k}\left(1 - \frac{s}{l+1}\right)\right) + \sum_{k=3}^{S+2} \frac{1}{4} \left(1 - \frac{\nu}{2S}\right) \frac{\binom{S-1}{k-3}}{\binom{DS}{k}} \kappa_l(s,k) \left(\frac{1}{l+1} + \frac{2}{k}\left(1 - \frac{s}{l+1}\right)\right) + \sum_{k=4}^{S+3} \frac{1}{4} \left(1 - \frac{\nu}{S}\right) \frac{\binom{S-1}{k-4}}{\binom{DS}{k}} \left(\frac{1}{l+1} + \frac{2}{k}\left(1 - \frac{s}{l+1}\right)\right) \kappa_l(s,k) \frac{1}{4} \right\}, \quad (10)$$

whereby Δ_3 is given by (3). With these Δ_i we obtain

$$d_l(1,1) = \sum_{i=1}^{5} \Delta_i. \quad (11)$$

Following the same principles we can calculate $d_l(k,j)$ for $k \geq 2$, which are increasingly complex expressions.