On the Multicast Capacity of Unidirectional and Bidirectional Packet-Switched WDM Ring Networks

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***Abstract*— In this paper we examine the relationship between the effective capacity (stability limit) of unidirectional and bidirectional packet-switched wavelength division multiplexing (WDM) ring networks for multicast traffic. We consider both bidirectional rings with one packet copy transmission per wavelength channel and two packet copy transmissions. We first prove bounds for the ratio of the multicast capacity of the bidirectional ring to the multicast capacity of the unidirectional ring. Specifically, we show that this ratio is at least two for two copy transmission in the bidirectional ring, and at most two for one copy transmission. We derive closed form expressions of the multicast capacity ratios for networks with a large number of nodes and from these expressions show that the ratios tend to two for a large number of multicast destinations. We demonstrate that for the bidirectional ring with two copy transmission the ratio becomes as large as 2.276. We also find that in the bidirectional ring, the capacity gain with two copy transmission over one copy transmission reaches 30.4%.**

***Index Terms*— Destination stripping, multicast, ring network, spatial wavelength reuse, throughput capacity, wavelength division multiplexing.**

I. INTRODUCTION

PACKET-switched ring wavelength division multiplexing (WDM) networks have been emerging in recent years as a promising solution to alleviate the capacity shortage in the metropolitan area, which is commonly referred to as metro gap. Packet-switched ring networks, such as the Resilient Packet Ring (RPR [1], overcome many of the shortcomings of circuit-switched ring networks, such as low provisioning flexibility for packet data traffic [2]. In these packet-switched ring networks, the destination nodes typically remove (strip) the packets destined to them from the ring. This *destination stripping* allows the destination node as well as other nodes downstream to utilize the wavelength channel for their own transmissions. With this so-called *spatial wavelength reuse*, multiple simultaneous transmissions can take place on any given wavelength channel.

The packet-switched ring WDM networks come in two main types: (i) unidirectional single-fiber rings, and (ii) bidirectional dual-fiber rings. In the unidirectional rings, the network nodes communicate over Λ, $\Lambda \geq 1$, wavelength channels that have the same light propagation direction. In the bidirectional ring, the nodes communicate over Λ_b, $\Lambda_b \geq 1$, wavelength channels on which the light propagates in the clockwise direction, and Λ_b additional wavelength channels on which the light propagates in the counter clockwise direction. For a fair comparison of the two types we set throughout $\Lambda_b = \Lambda/2$, with this setting the same total number of wavelength channels are operated in each network type. We consider two transmission strategies in the bidirectional ring network: a two copy transmission strategy where the source node sends two multicast packet copies on a given wavelength channel so as to minimize the travelled hop distance by the packet copies, thus maximizing spatial wavelength reuse, and a one copy transmission strategy where the source node sends one multicast packet copy on a given wavelength channel in the direction that reaches all destinations on the wavelength with the smallest hop count. The one copy transmission strategy is generally sub-optimal since the largest gap among the multicast destinations on a wavelength can lie anywhere around the ring perimeter; in order to avoid traversing the largest gap generally two packet copy transmissions are required which approach the largest gap from opposite directions.

For unicast traffic, the uni- and bidirectional ring networks have been extensively studied, see for instance [3]–[11]. It was found—and we re-confirm in a corollary in this study—that for uniform unicast traffic, the bidirectional ring network can support in the long run average exactly twice the number simultaneous transmissions than the unidirectional ring. Intuitively this is because in the unidirectional ring a packet travels on average halfway around the ring to reach its destination under the assumption of uniform traffic. In the bidirectional ring, the packets are generally transmitted in the direction that provides the shortest path to the destination. With this shortest path routing a packet travels on average around a quarter of the ring to reach its destination.

In this study, we examine arbitrary mixes of uniform unicast, multicast, and broadcast traffic. Multi-destination traffic is expected to account for a substantial portion of the metro network traffic due to the increasing popularity of applications,

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such as content distribution, distributed computing, multi-site tele-conferencing [12], and interactive distance learning. These applications are expected to demand substantial bandwidths due to the trend to deliver video and multimedia content in the High-Definition Television (HDTV) format and the emergence of video formats with resolutions higher than HDTV for digital cinema and tele-immersion applications. In addition, virtual private networks that combine an organization's geographically distributed local area networks are expected to contribute significantly to the multicast traffic volume in the metro area. While there is presently scant quantitative information about the multicast traffic volume there is ample anecdotal evidence of the emerging significance of this traffic type [13]. As a result, multicasting has been identified as an important service in optical networks [14], [15] and has received significant attention in circuit-switched WDM mesh and ring networks as well as packet-switched single-hop star networks as detailed in Section I-A.

In this paper we consider multicasting in packet-switched WDM ring networks. For arbitrary mixes of uniform unicast, multicast, and broadcast traffic, we formally prove the following bounds for the ratio of the multicast capacity of the bidirectional ring to the multicast capacity of the unidirectional ring: (A) with two copy transmission in the bidirectional ring the ratio is *at least two*, whereas (B) with one copy transmission it is *at most two*. For networks with a large number of nodes we provide closed-form expressions for the ratios of bidirectional to unidirectional multicast capacity as well as the capacity gain achieved with two copy transmission over one copy transmission in bidirectional ring networks. Based on these closed-form expressions we prove that the multicast capacity ratio of the bidirectional ring (both with two and one copy transmission) to the unidirectional ring tends to two for a large number of multicast destinations. Furthermore, we demonstrate that the ratio of bidirectional (with two copy transmission) to unidirectional multicast capacity reaches $11,235/4,936 \approx 2.276$, which is attained for multicasts with six receivers in networks with a total of $\Lambda = 2$ wavelengths. On the other hand, the ratio drops as low as 5/3 for the bidirectional ring with one copy transmission. In bidirectional rings, the capacity gain due to two copy transmission reaches 30.4% for multicasts with three receivers in networks with a total of $\Lambda = 2$ wavelengths.

A. Related Work

Optical packet-switched WDM ring networks have been experimentally demonstrated, see for instance [11], [16], and studied for unicast traffic, see for instance [3]–[11]. Multicasting in packet-switched WDM ring networks has received relatively little attention to date [8], [17]. The photonics level issues involved in multicasting over ring WDM networks are explored in [18], while a node architecture suitable for multicasting is studied in [19]. The general network architecture and MAC protocol issues arising from multicasting in packet-switched WDM ring networks are addressed in [16], [20]. The fairness issues arising when transmitting a mix of unicast and multicast traffic in a ring WDM network are examined in [21].

The multicast capacity of a unidirectional ring network and a bidirectional ring network with one copy transmission have been analyzed in [22] while the multicast capacity of a bidirectional ring network with two copy transmission has been analyzed in [23]. In this present study we make two main original contributions over the existing studies [22], [23]. First, for arbitrary mixes of unicast, multicast, and broadcast traffic in networks with an arbitrary number of nodes, we compare the capacities of the unidirectional and bidirectional ring networks and derive bounds on the ratios of the capacities. The existing studies [22], [23] examined each of these different ring networks in isolation but did not address the fundamental question as to how their capacities relate. Second, we derive simple, closed-form characterizations of the multicast capacities of the different ring networks for a large number of network nodes. The existing studies [22], [23] focused on expressing the multicast capacities as functions of the number of network nodes. The derived functional expressions are too complex for an insightful and detailed analytical comparison, which is not attempted in [22], [23]. In contrast, the simple, closed-form capacity expressions for a large number of nodes derived in this study facilitate the capacity comparisons and form the basis for specifying the ranges of the capacity ratio values.

We note that multicasting in circuit-switched WDM rings, which are fundamentally different from the packet-switched networks considered in this paper, have been extensively examined in the literature. The scheduling of connections and cost-effective design of bidirectional WDM rings was addressed, for instance in [24]. Cost-effective traffic grooming approaches in WDM rings have been studied for instance in [25], [26]. The routing and wavelength assignment in reconfigurable bidirectional WDM rings with wavelength converters was examined in [27]. The wavelength assignment for multicasting in circuit-switched WDM ring networks has been studied in [28]–[33]. For unicast traffic, the throughputs achieved by different circuit-switched and packet-switched optical ring network architectures are compared in [34].

We finally note for completeness that there has been increasing research interest in recent years for multicasting in general mesh circuit-switched WDM networks, see e.g., [35]–[39], including their circuit-switching capacity [40]. Similarly, multicasting in packet-switched single-hop star WDM networks has been intensely investigated, see for instance [41]–[44]

II. Compared WDM Ring Networks

A. Network Architecture

We compare optical wavelength division multiplexing (WDM) ring networks which interconnect N network nodes. We compare a single-fiber ring network where the nodes are interconnected by one unidirectional fiber, as illustrated in Fig. 1, with a bidirectional dual-fiber ring network. In both networks we number (index) the nodes sequentially as $n = 1, 2, \ldots, N$ in the clockwise direction.

We suppose that there are a total of Λ wavelength channels in each network. In particular, we suppose that there is one set of wavelength channels $\{1, \ldots, \Lambda\}$ in the single-fiber ring network, and furthermore we assume without loss of generality that the propagation direction on the fiber is the clockwise direction. In the bidirectional dual-fiber ring

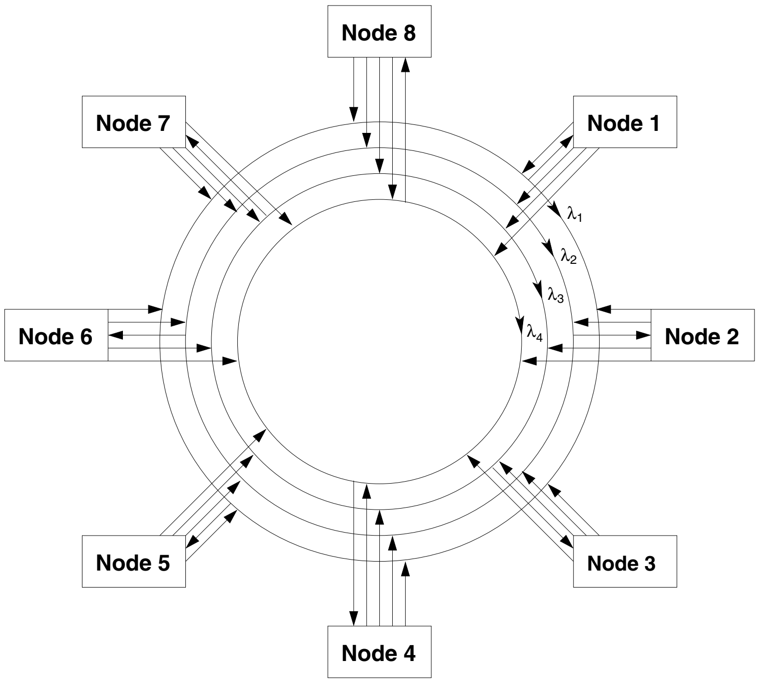

Fig. 1. Unidirectional WDM ring network connecting $N = 8$ nodes with $\Lambda = 4$ wavelength channels; each wavelength channel homes $\eta_u = N/\Lambda = 2$ nodes. In the corresponding bidirectional ring there are $\Lambda_b = \Lambda/2 = 2$ wavelength channels operated in each direction, with each wavelength channel homing $\eta_b = N/\Lambda_b = 4$ nodes.

network, $\Lambda_b = \Lambda/2$ wavelength channels, specifically the set of the wavelength channels $\{1, \ldots, \Lambda_b\}$, operate in the clockwise fiber direction, and an identical set $\{1, \ldots, \Lambda_b\}$ of wavelength channels operates in the counter clockwise fiber direction. We consider the family of node structures where each node (i) can transmit on any wavelength using either one or multiple tunable transmitters (TTs) or an array of Λ fixed-tuned transmitters (FTs), and (ii) receive on one wavelength using a single fixed-tuned receiver (FR), which is a widely considered node structure [3]–[11], [45]–[48].

For $N = \Lambda$ ($N = \Lambda_b$ in the bi-directional ring) each node has its own separate *home channel* for reception. For $N > \Lambda$ each wavelength is shared by several nodes for the reception of packets. We let $\eta_u = N/\Lambda$ denote the number of nodes that share a given wavelength as their home channel in the unidirectional ring, and assume that η_u is an integer. Correspondingly, we denote $\eta_b = N/\Lambda_b = 2\eta_u$ for the number of nodes that share a given wavelength as their home channel in the bidirectional ring. In particular, in the unidirectional ring the nodes $n = \lambda + k \cdot \Lambda$ with $k = 0, 1, \ldots, (\eta_u - 1)$ share the same drop wavelength (home channel) λ, $\lambda = 1, 2, \ldots, \Lambda$, i.e., have wavelength λ as their home channel. Analogously, in the bidirectional ring the nodes $n = \lambda + k \cdot \Lambda_b$ with $k = 0, 1, \ldots, (\eta_b - 1)$ share the same drop wavelength (home channel) λ, $\lambda = 1, 2, \ldots, \Lambda_b$. For brevity we will use the terminology that a node n *is on wavelength* λ if wavelength λ is the drop wavelength (home channel) of node n.

B. Traffic Model and Transmission Strategies

We consider traffic with fanout (number of destination nodes) F that is described by the distribution

$$\mu_l := P(F = l), \quad l = 1, \ldots, N - 1, \tag{1}$$

whereby $0 \leq \mu_l \leq 1$ and $\sum_{l=1}^{N-1} \mu_l = 1$. This fanout model accommodates arbitrary mixes of unicast, multicast, and broadcast traffic. Throughout we assume that a multicast is not sent to the source node, hence the maximum fanout is $N - 1$. As is common for capacity evaluations, we consider uniform traffic generation, i.e., all N nodes generate equivalent amounts of traffic. We consider uniform traffic destinations, i.e., the fanout set (set of destination nodes) for a given multicast with given fanout $F = l$ is drawn uniformly randomly from among the other $N - 1$ nodes. While our analysis assumes that the source node, the fanout, and the fanout set are drawn independently at random, this independence assumption is not critical for the analysis. Our results hold also for traffic patterns displaying correlations, as long as the long run average segment utilizations are equivalent to the utilizations with the independence assumption. For instance, our results hold for a correlated traffic model where a given source node transmits with a probability $p < 1$ to exactly the same set of destinations as the previous packet sent by the node, and with probability $1 - p$ to a new set of destination nodes drawn independently at random. We also note that uniform traffic is a reasonable traffic model for metro core ring networks. These core ring networks interconnect several metro edge ring networks and typically experience any-to-any traffic between all attached nodes [2].

To transmit a multicast packet, the source node generates a copy of the multicast packet for each wavelength that is the drop wavelength for at least one of the multicast destination nodes. In the unidirectional ring, the multicast packet copy is forwarded in the propagation direction on a given wavelength until the multicast packet copy reaches the last destination node on the wavelength; that node takes the packet off the ring. For an illustrative example, consider the network depicted in Fig. 1 and suppose node 8 is the source node and has a multicast packet destined to nodes 1, 2, 5, and 6. In the unidirectional ring network, a copy of the multicast packet travels on wavelength 1 in the clockwise direction from node 8 to node 5, and another copy travels on wavelength 2 from node 8 to node 6.

In the bidirectional ring network we distinguish the two copy and one copy transmission strategies for the multicast packet copy transmission on a given wavelength. With the two copy transmission strategy we consider the set of nodes containing the source node of the multicast and all the multicast destination nodes on the considered wavelength. Then we find the largest gap between any two neighboring nodes in the considered set. The multicast is served by (i) sending one copy of the multicast packet in the clockwise direction from the source node to the destination node that borders to the largest gap, and (ii) sending another copy of the multicast packet from the source node in the counter clockwise direction to the node bordering on the largest gap. (If the largest gap borders on the source node, then only one copy of the multicast packet is transmitted, namely in the direction opposite of the largest gap.) In the bidirectional ring network corresponding to the unidirectional ring network depicted in Fig. 1 there are $\Lambda_b = 2$ wavelengths in each ring direction with wavelength 1 homing nodes 1, 3, 5, and 7, while nodes 2, 4, 6, and 8 are homed on wavelength 2. Continuing the illustrative example from above, the largest gap on wavelength 1 is between nodes 1 and 5. Consequently, on the clockwise wavelength 1, one packet copy travels from node 8 to node 1, and another copy travels on the counter clockwise wavelength 1 from node 8 to node 5. Similarly, one packet copy travels on the clockwise wavelength 2 from node 8 to node 2, and another copy travels on the counter clockwise wavelength 2 from node 8 to node 6.

The one copy transmission strategy in the bidirectional ring transmits only one multicast packet copy per wavelength, namely in the direction that reaches all the multicast destinations on the wavelength with the smallest hop distance. We define the hop distance that a given multicast packet copy travels on a given wavelength λ as the number of nodes that the packet copy visits, whereby each traversed node (irrespective of whether the node is on the wavelength λ or a different wavelength) as well as the last destination node on the wavelength counts as a visited node. In the illustrative example, one packet copy travels on clockwise wavelength 1 from node 8 to node 5. Destination nodes 2 and 6 can be reached with six hops on both the clockwise and the counter clockwise wavelength 2 and the source node 8 selects either of the two with probability one half.

When a node receives a packet, it checks if there are additional destinations downstream; if so, it forwards the packet to the other destinations; otherwise, the node is the last destination and removes the packet from the ring. With this destination release (stripping), wavelengths can be spatially reused by downstream nodes, leading to an increased network capacity.

C. Capacity Definition

Toward the definition of the multicast capacity we first introduce the following terminology. We refer to the part of a wavelength channel between two successive nodes as *wavelength channel segment*, or in short as *segment*. A given wavelength channel consists of N segments for a total of $N\Lambda$ segments in both the uni- and the bidirectional ring. Since all nodes transmit traffic onto a given wavelength channel, but only the nodes homed on the channel remove traffic, the segments are typically non-uniformly loaded. In particular, the segments leading directly to the nodes homed on a channel are the most heavily utilized segments. We refer to these segments that attain the maximum utilization as *critical segments*. In the case of the unidirectional ring network depicted in Fig. 1, the critical segments on wavelength 1 are the segments between nodes 8 and 1, as well as the segment between nodes 4 and 5. The critical segments on wavelength 4 are the segment between nodes 3 and 4, as well as the segment between nodes 7 and 8. We denote $u^u_{\max}$, $u^b_{\max}$, and $u^{b1c}_{\max}$ for the maximum segment utilization in the unidirectional ring, the bidirectional ring with two copy transmission, and the bidirectional ring with one copy transmission, respectively. In the unidirectional ring there are η_u critical segments on each wavelength channel for a total of $\Lambda\eta_u$ critical segments in the unidirectional ring network. In the bidirectional ring, on the other hand, there are $\eta_b = 2\eta_u$ critical segments on each wavelength channel for a total of $2\Lambda\eta_u$ critical segments in the bidirectional ring network.

We consider the effective multicast capacity, which gives the maximum mean number of multicasts (stability limit) that can simultaneously take place in the network[1]. The effective multicast capacity, which we refer to henceforth as multicast capacity for brevity, is limited by the utilization of the critical segments. In particular, the multicast capacity is the reciprocal of the utilization of the critical segments. We denote $C^u_M = 1/u^u_{\max}$ for the multicast capacity of the unidirectional ring network, $C^b_M = 1/u^b_{\max}$ for the multicast capacity of the bidirectional ring network with two copy transmission, and $C^{b1c}_M = 1/u^{b1c}_{\max}$ for the multicast capacity of the bidirectional ring with one copy transmission.

III. Bound on Multicast Capacity Ratio C^b_M/C^u_M

Without loss of generality we consider the transmission of a multicast packet by node N, which is homed on wavelength Λ, to the destination nodes of the multicast on wavelength λ, $\lambda = 1, \ldots, \Lambda$, and suppose there are m multicast destinations on wavelength λ. We distinguish the two cases 1) $\lambda \neq \Lambda$, i.e., the destination nodes are on a different wavelength from the source node, and 2) $\lambda = \Lambda$, i.e., the destination nodes are on the same wavelength as the source node.

A. Case 1) $\lambda \neq \Lambda$

We focus initially on case 1) $\lambda \neq \Lambda$, in which there are in the bidirectional ring $m \leq \eta_b$ destination nodes on the considered wavelength. In particular, the potential destination nodes that can be reached with multicast packet copy transmission(s) on wavelength λ in the bidirectional ring are the nodes

[1] We note that [22] introduces both a nominal multicast capacity and an effective multicast capacity. The nominal capacity considers the hop distances required to serve multicasts, whereas the effective capacity considers the utilization of ring segments. Throughout the present study we consider the effective multicast capacity, which represents the stability limit of the network.

$n = \lambda + k \cdot \Lambda_b$ with $k = 0, 1, 2, 3 \ldots, (\eta_b - 1)$. Now consider these same nodes in the unidirectional ring. They are homed on the wavelengths λ and $\lambda + \Lambda_b$. In particular, wavelength λ homes the nodes $n = \lambda + k \cdot \Lambda_b$ with $k = 0, 2, \ldots, (\eta_b - 2)$ and wavelength $\lambda + \Lambda_b$ homes the nodes $n = \lambda + k \cdot \Lambda_b$ with $k = 1, 3, \ldots, (\eta_b - 1)$.

For convenience we renumber the nodes $n = \lambda + k \cdot \Lambda_b,\ k = 0, 1, 2, 3, \ldots, (\eta_b - 1)$ such that they have sequential indices, i.e., we renumber these nodes as $q = 1, 2, 3, 4, \ldots, \eta_b$. Note that in the bidirectional ring all these nodes are homed on wavelength λ. In the unidirectional ring the odd numbered nodes $q = 1, 3, \ldots, \eta_b - 1$ are homed on wavelength λ, and the even numbered nodes $q = 2, 4, \ldots, \eta_b$ are homed on wavelength $\lambda + \Lambda_b$. We then consider the following (cardinality preserving) bijective mapping f of the node indices $q = 1, 2, 3, \ldots, \eta_b$:

$$\begin{array}{rcl} 1 & \rightarrow & 1 \\ 2 & \rightarrow & \eta_b \\ 3 & \rightarrow & 2 \\ 4 & \rightarrow & \eta_b - 1 \\ \vdots & & \vdots \\ \eta_b - 1 & \rightarrow & \dfrac{\eta_b}{2} \\ \eta_b & \rightarrow & \dfrac{\eta_b}{2} + 1. \end{array} \quad (2)$$

For a subset $\mathcal{D} \subseteq \{1, 2, 3, \ldots, \eta_b\}$ of (destination) nodes let $f(\mathcal{D})$ denote the set $\{f(x),\ x \in \mathcal{D}\}$. Every subset of nodes is mapped to another set of the same cardinality.

Lemma 1: Suppose in the unidirectional ring, r critical segments are traversed to reach all nodes in $\mathcal{D}$, then the number of critical segments traversed in the bidirectional ring with two copy transmission to reach all nodes in $f(\mathcal{D})$ is *at most* r.

Proof: Let o denote the largest odd number (node index) in $\mathcal{D}$, and define $o := -1$ if there is no odd number in $\mathcal{D}$. Similarly, let e denote the largest even number in $\mathcal{D}$, and define $e := 0$ if there is no even number in $\mathcal{D}$. In order to reach all nodes in $\mathcal{D}$ in the unidirectional ring, $(o+1)/2 + e/2$ critical segments are traversed. In particular, $(o+1)/2$ critical segments are traversed on wavelength λ to reach the odd numbered nodes in $\mathcal{D}$ and $e/2$ critical segments are traversed on wavelength $\lambda + \Lambda_b$ in order to reach the even numbered nodes in $\mathcal{D}$. The mapping of $\mathcal{D}$ is given by

$$f(\mathcal{D}) \subseteq \{f(1),\ f(3), \ldots, f(o);\ f(2),\ f(4), \ldots, f(e)\} \quad (3)$$
$$= \{1,\ 2, \ldots, \frac{o+1}{2};\ \eta_b,\ \eta_b - 1, \ldots, \eta_b - \frac{e}{2} + 1\}. \quad (4)$$

This set of nodes is reached in the bidirectional ring with two copy transmission by traversing at most $e/2 + (o+1)/2$ critical segments. In particular, the nodes can be reached by transmitting the multicast copy in the clockwise direction to node $(o+1)/2$ (provided there is at least one odd numbered node) and in the counter clockwise direction to node $\eta_b - e/2 + 1$; possibly the nodes can be reached by traversing fewer critical segments. ■

For an example of a multicast that can be served in the bidirectional ring with two copy transmission while traversing less than $e/2 + (o+1)/2$ critical segments, consider a multicast with the destination nodes $\mathcal{D} = \{\eta_b - 1, \eta_b\}$, i.e., with $o = \eta_b - 1$ and $e = \eta_b$. In the unidirectional ring, $(o+1)/2 + e/2 = (\eta_b - 1 + 1)/2 + \eta_b/2 = \eta_b$ critical segments must be traversed to reach the two destination nodes. In the bidirectional ring, the two mapped destinations $f(\mathcal{D}) = \{\eta_b/2, \eta_b/2 + 1\}$ can be reached by transmitting one packet copy in the clockwise direction to node $\eta_b/2 + 1$, which requires only the traversal of $\eta_b/2 + 1$ critical segments.

For unicast traffic, for which there is $m = 1$ destination node on the considered wavelength (i.e., $\mathcal{D}$ has one element), and for broadcast traffic, for which there are $m = \eta_b$ destinations on the wavelength (i.e., $\mathcal{D} = \{1, 2, 3, \ldots, \eta_b\}$), the number of critical segments that need to be traversed in the bidirectional ring (irrespective of whether two or one copy transmission is employed) is exactly equal to the number required in the unidirectional ring.

Corollary 1: Suppose in the unidirectional ring, r critical segments are traversed to reach the destination node of a unicast or all the destination nodes of a broadcast, then the number of critical segments traversed in the bidirectional ring to reach the destination node of the unicast or all the destination nodes of the broadcast is *exactly* r.

Proof: In the case of a unicast, the single destination node has either an odd index o or even index e, which is mapped into a corresponding index $f(o)$ and served by a transmission in the clockwise direction or mapped into the corresponding index $f(e)$ and served by a transmission in the counter clockwise direction. In both the unidirectional ring and the bidirectional ring the number of traversed critical segments is $(o+1)/2$ in case of a odd destination node index or $e/2$ in case of an even destination node index.

In case of a broadcast, we have $o = \eta_b - 1$ and $e = \eta_b$. In both the uni- and bidirectional ring, $(o+1)/2 + e/2 = \eta_b$ critical segments are traversed to reach all destination nodes. ■

B. Case 2) $\lambda = \Lambda$

In the case 2) $\lambda = \Lambda$ we renumber the nodes $n = k \cdot \Lambda_b,\ \ k = 1, 2, \ldots, \eta_b - 1$ as $q = 1, 2, \ldots, \eta_b - 1$ and consider the bijective mapping f:

$$\begin{array}{rcl} 1 & \rightarrow & 1 \\ 2 & \rightarrow & \eta_b - 1 \\ 3 & \rightarrow & 2 \\ 4 & \rightarrow & \eta_b - 2 \\ \vdots & & \vdots \\ \eta_b - 2 & \rightarrow & \dfrac{\eta_b}{2} + 1 \\ \eta_b - 1 & \rightarrow & \dfrac{\eta_b}{2}. \end{array} \quad (5)$$

Similar to the notation for case 1) we denote $\mathcal{D} \subseteq \{1, 2, 3, \ldots, \eta_b - 1\}$, e for the largest even number in $\mathcal{D}$, and o for the largest odd number in $\mathcal{D}$. The lemma statement and proof are analogous to case 1). In particular, in the unidirectional ring the nodes in $\mathcal{D}$ are reached by traversing $e/2 + (o+1)/2$ critical segments. The nodes in the mapped set

$$f(\mathcal{D}) \subseteq \{f(1),\ f(3), \ldots, f(o);\ f(2),\ f(4), \ldots, f(e)\} \quad (6)$$
$$= \{1,\ 2, \ldots, \frac{o+1}{2};\ \eta_b,\ \eta_b - 1, \ldots, \eta_b - \frac{e}{2}\} \quad (7)$$

are reached in the bidirectional ring by traversing at most $e/2 + (o+1)/2$ critical segments.

The result proven in Lemma 1 leads us to the following bound on the multicast capacity ratio C_M^b/C_M^u.

Theorem 1: The multicast capacity of the bidirectional ring with two copy transmission C_M^b is at least twice the multicast capacity of the unidirectional ring C_M^u, i.e., $C_M^b/C_M^u \geq 2$.

Proof: Lemma 1 states that there is a one-to-one correspondence between the sets of multicast destination nodes in the unidirectional ring network and the bidirectional ring network with the following property: The number of traversed critical segments for a given realization of the set of multicast destination nodes in the bidirectional ring network is no larger than the number of traversed critical segments for the corresponding realization of the set of multicast destination nodes in the unidirectional ring network. This correspondence between the sets of multicast destinations in the unidirectional and bidirectional ring networks is given through the bijective mappings f given by (2) and (5). Note that for the considered uniform traffic model each of the corresponding sets of multicast destinations in the unidirectional and bidirectional ring networks has the same probability of being realized. Also, recall from Section II that there are twice as many critical segments in the bidirectional ring compared to the unidirectional ring. Hence, the mean utilization of a critical segment in the bidirectional ring is at most half as large as in the unidirectional ring. As a consequence the multicast capacity, which is the reciprocal of the utilization of the critical segments, is at least twice as large in the bidirectional ring compared to the unidirectional ring. ∎

IV. Bound on Multicast Capacity Ratio C_M^{b1c}/C_M^u

As in the preceding section, we consider without loss of generality a multicast from source node N, which is homed on wavelength Λ. As above the potential multicast destinations are in the set $\mathcal{D} \subseteq \{1, \ldots, \eta_b\}$ in case $\lambda \neq \Lambda$ and correspondingly in the set $\mathcal{D} \subseteq \{1, \ldots, \eta_b - 1\}$ in case $\lambda = \Lambda$.

A. *Case 1) $\lambda \neq \Lambda$*

Lemma 2: Suppose in the unidirectional ring, r critical segments are traversed to reach all nodes in $\mathcal{D}$, then the number of critical segments traversed in the bidirectional ring with one copy transmission to reach all nodes in $f(\mathcal{D})$, whereby the bijective mapping f given by (2), is *at least* r.

Proof: Let o denote the largest odd number (node index) in $\mathcal{D}$, and define $o := -1$ if there is no odd number in $\mathcal{D}$. Similarly, let e denote the largest even number in $\mathcal{D}$, and define $e := 0$ if there is no even number in $\mathcal{D}$.

If $o = -1$ or $e = 0$, then the number of traversed critical segments to reach the nodes in $\mathcal{D}$ in the unidirectional ring is exactly equal to the number of traversed critical segments to reach the nodes in $f(\mathcal{D})$ (with $f(\cdot)$ given by (2)) in the bidirectional ring.

Now consider $o \geq 1$ and $e \geq 2$. In this case $(o+1)/2 + e/2$ critical segments are traversed in the unidirectional ring. Let $g(\cdot)$ denote the bijective mapping $g(j) := \eta_b + 1 - j,\ j \in \{1, \ldots, \eta_b\}$. Then it follows that the mapping $f \circ g : \{1, \ldots, \eta_b\} \rightarrow \{1, \ldots, \eta_b\}$ is bijective, and when $\mathcal{D}$ contains both odd and even numbers, then $f \circ g(\mathcal{D})$ contains both numbers that are smaller than $(\eta_b + 1)/2$ and numbers that are larger than $(\eta_b + 1)/2$. Since o is the largest odd number in $\mathcal{D}$, $g(o)$ is the smallest even number in $g(\mathcal{D})$. Similarly, since e is the largest even number in $\mathcal{D}$, $g(e)$ is the smallest odd number in $g(\mathcal{D})$. Hence, the number of traversed critical segments in the bidirectional ring which reaches all multicast destinations on a wavelength by transmitting one copy of the multicast packet in the direction that reaches all the destination nodes $f(g(\mathcal{D}))$ homed on the wavelength with the shortest hop count is given by

$$\left(\eta_b + 1 - \frac{g(o)}{2}\right) \wedge \left(\eta_b - \frac{g(e) - 1}{2}\right) \quad (8)$$
$$= \left(\eta_b - \frac{\eta_b - o - 1}{2}\right) \wedge \left(\eta_b - \frac{\eta_b - e}{2}\right) \quad (9)$$
$$= \left(\frac{\eta_b}{2} + \frac{o+1}{2}\right) \wedge \left(\frac{\eta_b}{2} + \frac{e}{2}\right) \quad (10)$$
$$= \frac{\eta_b}{2} + \frac{(o+1) \wedge e}{2} \quad (11)$$
$$\geq \frac{(o+1) \vee e}{2} + \frac{(o+1) \wedge e}{2} \quad (12)$$
$$= \frac{(o+1)}{2} + \frac{e}{2}, \quad (13)$$

where we denote $x \wedge y := \min(x, y)$ and $x \vee y := \max(x, y)$ and (12) follows by noting that $\eta_b/2$ is larger than or equal to both $e/2$ and $(o+1)/2$. ∎

B. *Case 2) $\lambda = \Lambda$*

The lemma statement and proof in this case are analogous to the statement and proof in the preceding section. We employ the mappings $f(\cdot)$ defined in (5) and $g(j) := \eta_b - j,\ j \in \{1, \ldots, \eta_b - 1\}$. As before we denote $\mathcal{D} \subseteq \{1, \ldots, \eta_b - 1\}$ for the set of potential destination nodes as well as $o,\ o = -1, 1, \ldots, \eta_b - 1$, for the largest odd and $e,\ e = 0, 2, \ldots, \eta_b - 2$, for the largest even number in $\mathcal{D}$. The case when $o = -1$ or $e = 0$ is the same as in the preceding section.

When $o \geq 1$ and $e \geq 2$ then $(o+1)/2 + e/2$ critical segments are traversed in the unidirectional ring. Since o is the largest odd number in $\mathcal{D}$, $g(o)$ is the smallest even number in $g(\mathcal{D})$. Similarly, $g(e)$ is the smallest odd number in $g(\mathcal{D})$. Hence, the number of traversed critical segments in the bidirectional ring is

$$\left(\eta_b - \frac{g(o) + 1}{2}\right) \wedge \left(\eta_b - \frac{g(e)}{2}\right) \quad (14)$$
$$= \left(\eta_b - \frac{\eta_b + 1 - o}{2}\right) \wedge \left(\eta_b - \frac{\eta_b - e}{2}\right) \quad (15)$$
$$= \frac{\eta_b + [(o-1) \wedge e]}{2} \quad (16)$$

$$\geq \frac{2+[(o-1)\vee e]+[(o-1)\wedge e]}{2} \quad (17)$$
$$= \frac{2+o-1+e}{2} \quad (18)$$
$$= \frac{(o+1)}{2}+\frac{e}{2}, \quad (19)$$

where the inequality (17) follows by noting that $o \leq \eta_b - 1$ and $e \leq \eta_b - 2$.

Based on Lemma 2 we obtain with a proof that mirrors that of Theorem 1 the following bound on the multicast capacity ratio C_M^{b1c}/C_M^u.

Theorem 2: The multicast capacity of the bidirectional ring with one copy transmission C_M^{b1c} is at most twice the multicast capacity of the unidirectional ring C_M^u, i.e., $C_M^{b1c}/C_M^u \leq 2$.

For bidirectional ring networks, we immediately obtain from Theorems 1 and 2 that the multicast capacity with two copy transmission is larger or equal to the capacity with one copy transmission, i.e., $C_M^b \geq C_M^{b1c}$. Furthermore, from Corollary 1 we have that $C_M^b = C_M^{b1c}$ for unicast traffic and broadcast traffic. Moreover, we directly obtain from Theorem 2 that $C_M^{b1c}/(2C_M^u) \leq 1 = C_M^b/C_M^b$. Consequently, we obtain a lower bound on the relative capacity increase obtained with two copy transmission over one copy transmission in terms of the capacities of the unidirectional ring and the bidirectional ring with two copy transmission as

$$\frac{C_M^b}{C_M^{b1c}} \geq \frac{C_M^b}{2C_M^u}. \quad (20)$$

Note that our analysis of the bounds on the multicast capacity ratios applies to uniform traffic, which is typical for metro core ring networks [2]. Metro edge ring networks that interconnect individual access networks with the metro core ring, on the other hand, experience typically non-uniform traffic. We remark that the derived capacity ratios do not hold for non-uniform traffic. For an illustrative counterexample consider rings with $N = 6$ nodes and $\Lambda_b = 1$ and suppose that only one (hotspot) node sends unicast traffic uniformly to all other nodes. Then we obtain $C_M^b = C_M^{b1c} = 2$ and $C_M^u = 5/3$. In general, the study of non-uniform traffic in packet-switched ring networks is largely an open area for future research and we believe the present analysis for uniform traffic is an important stepping stone toward examining that open area. For an initial investigation of non-uniform traffic with a single hotspot in a single-wavelength bidirectional packet-switched ring we refer the interested reader to [49].

V. Multicast Capacity Ratios for Large Number of Nodes N

In this section we consider the case of uniform traffic for a large number of network nodes $N \to \infty$, and keep the number of wavelength channels Λ fixed (independent of N). We derive relatively simple, closed-form characterizations of the multicast capacity ratios, which allow for an insightful study of the ratios. We first derive characterizations of the maximum segment utilization and multicast capacity for each of the considered networks, and subsequently examine the resulting multicast ratios.

Formally, let $\mu_l^N,\ l = 1, \ldots, N-1$, denote the multicast fanout distribution and suppose that this fanout distribution has a nondegenerate limit distribution $\mu_l,\ l \geq 1$, that satisfies, $\lim_{N\to\infty} \mu_l^N = \mu_l,\ l \geq 1$, as well as $\mu_l \geq 0$ and $\sum_{l=1}^{\infty} \mu_l = 1$. (The case of a degenerate "escaping" fanout mass is considered in Appendix B.) Throughout this section we adopt the following notational convention: Notations with the superscript N refer to quantities in a network with a specific number of nodes N, whereas notations without the superscript N refer to quantities in a network with $N \to \infty$.

A. Maximum Segment Utilization and Multicast Capacity of Unidirectional Ring

Let $u_{\max}^{u,N}$ denote the largest segment utilization (probability) on the unidirectional ring with N nodes and note that $C_M^{u,N} = 1/u_{\max}^{u,N}$ is the corresponding effective multicast capacity. Note that $u_{\max}^{u,N}$ is given by the probability that a fixed critical segment is utilized by a sender (node) that is uniformly randomly chosen from among the N nodes of the ring. This probability coincides with the probability that a fixed sender (node) utilizes a critical segment which is uniformly randomly chosen from among the set of all critical segments. We pursue the evaluation of this latter probability. Without loss of generality we suppose node N to be the sender. The (random) critical segment is given by $W_u + K_u\Lambda$, where W_u is a uniform random variable on the set $\{1, \ldots, \Lambda\}$ of all wavelengths and K_u is a uniform random variable on the set $\{0, \ldots, \eta_u - 1\}$ being independent of W_u. Denoting by $M(N; \lambda, k)$ the event that a multicast generated by node N utilizes segment $\lambda + k\Lambda$ on wavelength λ we obtain (21) through (24) (shown on the next page).

Now letting $S_\lambda^{u,N}$ be a random variable denoting the number of critical segments that are utilized in a network with N nodes on wavelength λ during the transmission of the multicast generated by node N to all of its destination nodes on λ, we have $\mathbb{P}[M(N; \lambda, k)] = \mathbb{P}[S_\lambda^u \geq k+1]$ and thus (25) and (26) (shown on the next page). Conditioning on the number of multicast destinations l we obtain

$$u_{\max}^{u,N} = \frac{1}{N}\sum_{\lambda=1}^{\Lambda}\sum_{l=1}^{N-1} \mathbb{E}[S_\lambda^{u,N}|F = l]\ \mu_l^N \quad (27)$$
$$= \sum_{l=1}^{N-1} u_{\max}^{u,N}(l)\ \mu_l^N, \quad (28)$$

with

$$u_{\max}^{u,N}(l) := \frac{1}{N}\sum_{\lambda=1}^{\Lambda} \mathbb{E}[S_\lambda^{u,N}|F = l] \quad (29)$$

denoting the largest utilization probability with fixed fanout $F = l$.

We proceed to derive the utilization probability $u_{\max}^{u,N}(l)$ in the limit for a large number of nodes $N \to \infty$ and denote $u_{\max}^u(l) = \lim_{N\to\infty} u_{\max}^{u,N}(l)$. Toward this goal we rewrite (29) with (i) $R_\lambda^{u,N} := S_\lambda^{u,N}/\eta_u$ being a random variable that denotes the relative number of traversed critical segments on λ, and (ii) $\mathbb{P}(F_\lambda = m|F = l)$ denoting the conditional probability for having $F_\lambda = m$ destinations on wavelength λ

$$\begin{aligned} u_{\max}^{u,N} &= \mathbb{P}[M(N;W_u,K_u)] && (21)\\ &= \sum_{\lambda=1}^{\Lambda}\sum_{k=0}^{\eta_u-1}\mathbb{P}[M(N;W_u,K_u)|W_u=\lambda,K_u=k]\;\mathbb{P}[W_u=\lambda,K_u=k] && (22)\\ &= \sum_{\lambda=1}^{\Lambda}\sum_{k=0}^{\eta_u-1}\mathbb{P}[M(N;\lambda,k)]\;\frac{1}{\Lambda}\frac{1}{\eta_u} && (23)\\ &= \frac{1}{N}\sum_{\lambda=1}^{\Lambda}\sum_{k=0}^{\eta_u-1}\mathbb{P}[M(N;\lambda,k)] && (24) \end{aligned}$$

$$\begin{aligned} u_{\max}^{u,N} &= \frac{1}{N}\sum_{\lambda=1}^{\Lambda}\sum_{k=0}^{\eta_u-1}\mathbb{P}[S_\lambda^{u,N}\geq k+1] = \frac{1}{N}\sum_{\lambda=1}^{\Lambda}\sum_{k=1}^{\eta_u}\mathbb{P}[S_\lambda^{u,N}\geq k] && (25)\\ &= \frac{1}{N}\sum_{\lambda=1}^{\Lambda}\mathbb{E}[S_\lambda^{u,N}] && (26) \end{aligned}$$

given a total of $F=l$ multicast destinations to obtain

$$u_{\max}^{u,N}(l) = \frac{1}{\Lambda}\sum_{\lambda=1}^{\Lambda}\sum_{m=0}^{l}\mathbb{E}[R_\lambda^{u,N}|F_\lambda=m]\cdot\mathbb{P}(F_\lambda=m|F=l). \quad (30)$$

Now we observe the following two main points in the limit $N\to\infty$:

1) For $\lambda = 1,\ldots,\Lambda-1$ the random variable $R_\lambda^{u,N}$ may take on the values $0,\frac{1}{\eta_u},\frac{2}{\eta_u},\ldots,\frac{\eta_u-1}{\eta_u},\frac{\eta_u}{\eta_u}=1$, whereby $R_\lambda^{u,N} := 0$ if there is no destination node on λ. For $m>0$ the destinations on λ are uniformly randomly distributed (without resampling) on the nodes homed on λ which correspond to the relative distances $\{\frac{1}{\eta_u},\frac{2}{\eta_u},\ldots,\frac{\eta_u-1}{\eta_u},\frac{\eta_u}{\eta_u}\}$ (or distance 0 if $m=0$). The largest (index) of those destinations provides the realization of $R_\lambda^{u,N}$. In particular, the distribution of $R_\lambda^{u,N}$ given that there are $m>0$ destinations on λ coincides with the distribution of the maximum (index) of m destinations that are chosen randomly (without resampling) from among $\{\frac{1}{\eta_u},\frac{2}{\eta_u},\ldots,\frac{\eta_u-1}{\eta_u},\frac{\eta_u}{\eta_u}\}$. The larger N, the tighter these relative distances are fitted into the interval $[0,1]$. As $N\to\infty$ the random variable $R_\lambda^{u,N}$ approximates a random variable, R_λ^u, which may take on any value from $[0,1]$. In particular, the distribution of R_λ^u given that there are m destinations on λ coincides with the distribution of $T_{m:m}$ where $T_{1:m},\ldots,T_{m:m}$ is the so-called order statistics of m independently uniformly distributed random variables on $[0,1]$.[2] Hence, the expectation of R_λ^u given $F_\lambda=m$ is given by
$$\mathbb{E}[R_\lambda^u|F_\lambda=m]=\frac{m}{m+1}. \quad (31)$$
For $\lambda=\Lambda$ the setting considered above must be slightly modified since at most η_u-1 critical segments are traversed on the wavelength homing the source node N. We omit the details of this modification, which results in the limit $N\to\infty$ also in (31).

2) The hypergeometric distribution of $\mathbb{P}(F_\lambda=m|F=l)$ approaches a binomial distribution as $N\to\infty$. Specifically, the value of $\mathbb{P}(F_\lambda=m|F=l)$ is specified in (32) and (33) (shown on the next page), whereby we employ the definition $0^0=1$ to accommodate the case $\Lambda=1$.

Thus, overall (34), (35), (36), and (37) (shown on the next page) are true.

B. Maximum Segment Utilization and Multicast Capacity of Bidirectional Ring with Two Copy Transmission

Similar to the unidirectional case we let $u_{\max}^{b,N}$ denote the probability that a given critical segment is utilized by a sender (node) that is uniformly randomly chosen from among the N nodes of the ring, and note that the effective multicast capacity is given by $C_M^{b,N}=1/u_{\max}^{b,N}$. As above, we observe that $u_{\max}^{b,N}$ coincides with the probability that a fixed sender (node) utilizes a critical segment which is uniformly randomly chosen from among the set of all critical segments. To specify the latter probability we assume without loss of generality node N to be the sender. Let W_b be a uniform random variable on the set $\{1,\ldots,\Lambda_b\}$ of all wavelengths and K_b be a uniform random variable on the set $\{0,\ldots,\eta_b-1\}$ being independent of W_b. Denote by $M(N;\lambda,k;+)$, respectively $M(N;\lambda,k;-)$, the event that a multicast generated by node N utilizes segment $\lambda+k\Lambda_b$ on λ in direction "+", respectively segment $(N-\Lambda_b+1)+\lambda-k\Lambda_b$ on λ, $\lambda\neq\Lambda_b$ in direction "−". (The case $\lambda=\Lambda$ requires some adjustment for the − direction, which vanishes in the limit $N\to\infty$ as noted below.) Moreover, we write $S_{\lambda,+}^b$, $S_{\lambda,-}^b$ and S_λ^b for the random variables that denote the numbers of critical segments that are utilized on wavelength λ during the transmission in direction "+", in direction "−" and in either directions, respectively, of the multicast generated by node N to all of its destination nodes on λ. In particular, $S_\lambda^b=S_{\lambda,+}^b+S_{\lambda,-}^b$. We have

$$\begin{aligned} u_{\max}^{b,N} &= \mathbb{P}[M(N;W_b,K_b;+)] = \mathbb{P}[M(N;W_b,K_b;-)] && (38)\\ &= \frac{1}{2}\Big(\mathbb{P}[M(N;W_b,K_b;+)]+\mathbb{P}[M(N;W_b,K_b;-)]\Big). && (39) \end{aligned}$$

[2]Let $T_1,\ldots,T_m$ be m independent uniformly distributed random variables on $[0,1]$. The *order statistics* $T_{1:m},\ldots,T_{m:m}$ of $T_1,\ldots,T_m$ is defined to be a permutation of $T_1,\ldots,T_m$ such that $T_{1:m}\leq\cdots\leq T_{m:m}$.

$$\mathbb{P}(F_\lambda = m|F = l) = \frac{\binom{\eta_u}{m}\binom{N-1-\eta_u}{l-m}}{\binom{N-1}{l}} \tag{32}$$

$$\longrightarrow \binom{l}{m}\Big(\frac{1}{\Lambda}\Big)^m\Big(\frac{\Lambda-1}{\Lambda}\Big)^{l-m}, \qquad \text{as } N \to \infty \tag{33}$$

$$u^u_{\max}(l) = \lim_{N\to\infty}\frac{1}{\Lambda}\sum_{\lambda=1}^{\Lambda}\sum_{m=0}^{l}\mathbb{E}[R^{u,N}_\lambda|F_\lambda = m]\mathbb{P}(F_\lambda = m|F = l) \tag{34}$$

$$= \lim_{N\to\infty}\sum_{m=0}^{l}\mathbb{E}[R^{u,N}_1|F_1 = m]\mathbb{P}(F_1 = m|F = l) \tag{35}$$

$$= \sum_{m=1}^{l}\frac{m}{m+1}\binom{l}{m}\Big(\frac{1}{\Lambda}\Big)^m\Big(\frac{\Lambda-1}{\Lambda}\Big)^{l-m} \tag{36}$$

$$\lim_{N\to\infty} C^{u,N}_M = \frac{1}{\sum_{l=1}^{\infty}\Big(\sum_{m=1}^{l}\ \frac{m}{m+1}\ \binom{l}{m}\big(\frac{1}{\Lambda}\big)^m\big(\frac{\Lambda-1}{\Lambda}\big)^{l-m}\Big)\ \mu_l} \tag{37}$$

Proceeding analogously to the unidirectional ring analysis we obtain

$$u^{b,N}_{\max} = \frac{1}{2}\Big(\frac{1}{N}\sum_{\lambda=1}^{\Lambda_b}\mathbb{E}[S^b_{\lambda,+}] + \frac{1}{N}\sum_{\lambda=1}^{\Lambda_b}\mathbb{E}[S^b_{\lambda,-}]\Big) \tag{40}$$

$$= \frac{1}{2N}\sum_{\lambda=1}^{\Lambda_b}\mathbb{E}[S^b_\lambda] \tag{41}$$

$$= \sum_{l=1}^{N-1}u^{b,N}_{\max}(l)\ \mu^N_l \tag{42}$$

with

$$u^{b,N}_{\max}(l) = \frac{1}{2\Lambda_b}\sum_{\lambda=1}^{\Lambda_b}\sum_{m=0}^{l}\mathbb{E}[R^{b,N}_\lambda|F_\lambda = m]\cdot\mathbb{P}(F_\lambda = m|F = l), \tag{43}$$

where $R^{b,N}_\lambda := S^{b,N}_\lambda/\eta_b$ denotes the relative number of traversed critical segments on λ (in comparison to the number η_b of all critical segments on λ).

In the limit $N \to \infty$ the m, $m \geq 1$, destinations on λ, $\lambda = 1, \ldots, \Lambda_b$, are (independently) uniformly randomly chosen from the interval $[0, 1]$. The realization of R^b_λ is defined to be "1 − maximal gap between any two neighboring destination nodes (indices)". This reflects the two copy transmission mechanism according to which one copy of the multicast is sent in the clockwise direction to the destination node that borders to the largest gap, and another copy is sent in the counter clockwise direction to the destination node that borders to the largest gap. In particular, the distribution of R^b_λ given that there are m destinations on λ coincides with the distribution of "1 − largest gap in the order statistics $T_{1:m}, \ldots, T_{m:m}$ of m independently uniformly distributed random variables on $[0, 1]$". More precisely, the distribution of R^b_λ given that there are m destinations on λ coincides with the one of $1 - T^{\text{gap}}_{(m)}$, where $T^{\text{gap}}_{(m)} := \max\{T_{i:m} - T_{i-1:m} : i = 1, \ldots, m\} \vee (1 - T_{m:m})$, $T_{0:m} := 0$ and $a \vee b := \max\{a, b\}$. The distribution of $T^{\text{gap}}_{(m)}$ is given in [50, p. 81], from which

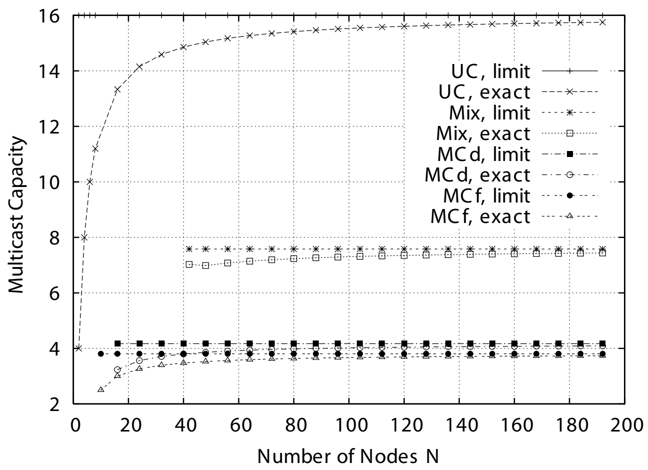

Fig. 2. Multicast capacity of bidirectional ring with two copy transmission: Comparison of exact $C^{b,N}_M$ with $\lim_{N\to\infty} C^{b,N}_M$; Λ_b = 2, fixed

we deduce the expectation of $T^{\text{gap}}_{(m)}$ as

$$1 - \mathbb{E}[R^b_\lambda|F_\lambda = m] = \mathbb{E}[T^{\text{gap}}_{(m)}] = \frac{1}{m+1}\sum_{i=1}^{m+1}\binom{m+1}{i}(-1)^{i+1}\frac{1}{i}. \tag{44}$$

We note that this expectation can be conveniently expressed in terms of the Digamma function $\Psi(x) := \frac{d}{dx}\log(\Gamma(x))$, $x > 0$, with Γ denoting the well-known Gamma function $\Gamma(x) := \int_0^\infty e^{-t}t^{x-1}dt$, $x > 0$, and the Euler-Mascheroni constant $\gamma \approx 0.5772156649$ as

$$\mathbb{E}[R^b_\lambda|F_\lambda = m] = 1 - \frac{1}{m+1}(\Psi(m+2) + \gamma). \tag{45}$$

Thus, we obtain (46) and (47) (shown on the next page).

To illustrate the convergence of the actual multicast capacity $C^{b,N}_M$ for a network with N nodes to the multicast capacity $\lim_{N\to\infty} C^{b,N}_M$ we compare in Fig. 2 the actual (exact) multicast capacity $C^{b,N}_M$ evaluated with the computational procedure derived in [23] with $\lim_{N\to\infty} C^{b,N}_M$ given in (47).

$$u^b_{\max}(l) = \frac{1}{2}\left\{1 - \sum_{m=0}^{l} \frac{1}{m+1}\left[\Psi(m+2)+\gamma\right]\binom{l}{m}\left(\frac{1}{\Lambda_b}\right)^m\left(\frac{\Lambda_b-1}{\Lambda_b}\right)^{l-m}\right\} \quad (46)$$

$$\lim_{N\to\infty} C_M^{b,N} = \frac{2}{1-\sum_{l=1}^{\infty}\left(\sum_{m=0}^{l} \frac{1}{m+1}\left\{\Psi(m+2)+\gamma\right\}\binom{l}{m}\left(\frac{1}{\Lambda_b}\right)^m\left(\frac{\Lambda_b-1}{\Lambda_b}\right)^{l-m}\right)\mu_l} \quad (47)$$

We consider a bidirectional ring with $\Lambda_b = 2$ wavelength channels in each direction and plot curves for unicast traffic with $\mu_1 = 1,\ \mu_l = 0$ for $l \geq 2$ (UC), multicast traffic with a fixed fanout with $\mu_8 = 1,\ \mu_l = 0$ for $l \neq 8$ (MCf), multicast traffic with a distributed fanout with $\mu_l = 1/15$ for $l = 1, \ldots, 15$ and $\mu_l = 0$ for $l \geq 16$ (MCd), and mixed traffic with $\mu_1 = 0.75$ and $\mu_l = 0.25/41$ for $l = 2, \ldots, 42$ and $\mu_l = 0$ for $l \geq 43$ (Mix). We observe from the figure that the exact multicast capacity quickly converges to the limit capacity $\lim_{N\to\infty} C_M^{b,N}$ as the number of nodes N increases. For N = 40 nodes, i.e., 20 nodes homed on a wavelength with the considered $\Lambda_b = 2$ wavelengths in each ring direction, the exact capacity is within less than 10% of the limit capacity. Comparing the Mix with the UC results, we observe that a moderate amount of 25% of multicast traffic cuts the capacity approximately in half compared to a network serving no multicast traffic, illustrating that multicast traffic has a very significant impact on the overall network performance.

C. Maximum Segment Utilization and Multicast Capacity of Bidirectional Ring with One Copy Transmission

The main difference of the one copy transmission from the two copy transmission analyzed in the preceding section is that with the one copy transmission the realization of the relative distance R_λ^{b1c} is defined to be the minimum of the largest destination node (index) and "1 − smallest destination node (index)". This reflects the one copy transmission mechanism according to which one packet copy is sent in the direction that minimizes the hop distance. In particular, the distribution of R_λ^{b1c} given that there are m destinations on $\lambda,\ \lambda = 1, \ldots, \Lambda_b$, coincides with the distribution of $T_{(m)} := \min\{T_{m:m}, 1 - T_{1:m}\}$, where $T_{1:m}, \ldots, T_{m:m}$ is the order statistics of m independently uniformly distributed random variables on $[0, 1]$. The distribution of $T_{(m)}$ is given by (48) through (52) (shown on the next page), where $\mathbf{1}_{y\geq 1/2} :=$ 1 for $y \geq 1/2$, and $\mathbf{1}_{y\geq 1/2} := 0$ otherwise. Then elementary analytic arguments yield $\mathbb{E}[R^b_\lambda | F_\lambda = m] = \mathbb{E}[T_{(m)}] = \frac{2m-1}{2(m+1)}$. Thus, we obtain

$$u^{b1c}_{\max}(l) = \frac{1}{2}\sum_{m=1}^{l} \frac{2m-1}{2(m+1)}\binom{l}{m}\left(\frac{1}{\Lambda_b}\right)^m\left(\frac{\Lambda_b-1}{\Lambda_b}\right)^{l-m} \quad (53)$$

and

$$\lim_{N\to\infty} C_M^{b1c,N} = \frac{2}{\sum_{l=1}^{\infty}\sum_{m=1}^{l} \frac{2m-1}{2(m+1)}\binom{l}{m}\left(\frac{1}{\Lambda_b}\right)^m\left(\frac{\Lambda_b-1}{\Lambda_b}\right)^{l-m}\mu_l}. \quad (54)$$

D. Multicast Capacity Ratios

From the analyses in the preceding three subsections we immediately obtain the following values for the bidirectional (with two and one copy transmission) to unidirectional multicast capacity ratios.

Theorem 3: See (55) (shown on the next page).

Theorem 4: See (56) (shown on the next page).

We employ the relatively simple closed-form expressions in (55) and (56) to further investigate the behaviors of the multicast ratios.

Letting $C_M^{u,N}(l)$, $C_M^{b,N}(l)$, and $C_M^{b1c,N}(l)$ refer to the multicast capacities of the unidirectional ring, the bidirectional ring with two copy transmission, and the bidirectional ring with one copy transmission, respectively, for a fixed fanout $F = l$ (that is, $\mu_l = 1$ and $\mu_{l'} = 0\ \forall\ l' \neq l$), we obtain the following corollaries, which are proven in Appendix A.

Corollary 2: $\lim_{l\to\infty}\left\{\lim_{N\to\infty}\frac{C_M^{b,N}(l)}{C_M^{u,N}(l)}\right\} = 2.$

Corollary 3: $\lim_{l\to\infty}\left\{\lim_{N\to\infty}\frac{C_M^{b1c,N}(l)}{C_M^{u,N}(l)}\right\} = 2.$

The intuitive explanation for the results in these corollaries is as follows. With an increasing number of multicast destinations l in a network with a large number of nodes N it becomes increasingly likely that (i) there are destinations on all home wavelengths ($\{1, \ldots, \Lambda\}$ in the unidirectional ring and $\{1, \ldots, \Lambda_b\}$ in the bidirectional ring), and (ii) that there are multicast destinations all around the ring perimeter on each wavelength requiring each packet copy to make essentially a full round trip around the ring. As a consequence, at most one multicast can take place at a time on the one set of wavelengths channels $\{1, \ldots, \Lambda\}$ in the unidirectional ring, whereas two simultaneous multicasts can take place on the two identical sets of wavelength channels $\{1, \ldots, \Lambda_b\}$ in the bidirectional ring.

In summary, we have found so far that the multicast capacity ratio C_M^b/C_M^u (with two copy transmission in the bidirectional ring) is bounded from below by two (see Theorem 1) and tends to two when there is a large number of multicast destinations (see Corollary 2). A remaining question to ask is how large can the multicast capacity ratio C_M^b/C_M^u get? Analogously, we have found that the multicast capacity ratio C_M^{b1c}/C_M^u (with one copy transmission in the bidirectional ring) is bounded from above by two (see Theorem 2) and tends to two for a large number of multicast destinations (see Corollary 3), but it remains to characterize how small this ratio can get.

Before we further pursue the question of the how large (small) the multicast capacity ratio with two (one) copy transmission in the bidirectional ring gets we plot the ratio $\lim_{N\to\infty} C_M^{b,N}/C_M^{u,N}$ in Fig. 3 to obtain further insights into its behavior.

$$\begin{aligned}
\mathbb{P}[T_{(m)} \le y] &= \mathbb{P}[T_{m:m} \le y \text{ or } T_{1:m} \ge 1-y] && (48)\\
&= \mathbb{P}[T_{m:m} \le y] + \mathbb{P}[T_{1:m} \ge 1-y] - \mathbb{P}[T_{m:m} \le y,\ T_{1:m} \ge 1-y] && (49)\\
&= \mathbb{P}[T_1 \le y, \dots, T_m \le y] + \mathbb{P}[T_1 \ge 1-y, \dots, T_m \ge 1-y]\\
&\quad -\mathbb{P}[T_1 \in [1-y, y], \dots, T_m \in [1-y, y]] && (50)\\
&= y^m + (1-(1-y))^m - (y-(1-y))^m\ \mathbf{1}_{y\ge 1/2} && (51)\\
&= 2\ y^m - (2y-1)^m\ \mathbf{1}_{y\ge 1/2} && (52)
\end{aligned}$$

$$\lim_{N\to\infty} \frac{C_M^{b,N}}{C_M^{u,N}} = 2\ \frac{\sum_{l=1}^{\infty}\left(\sum_{m=1}^{l}\ \frac{m}{m+1}\ \binom{l}{m}\left(\frac{1}{\Lambda}\right)^m\left(\frac{\Lambda-1}{\Lambda}\right)^{l-m}\right)\ \mu_l}{1-\sum_{l=1}^{\infty}\left(\sum_{m=0}^{l}\ \frac{1}{m+1}\left\{\Psi(m+2)+\gamma\right\}\binom{l}{m}\left(\frac{1}{\Lambda_b}\right)^m\left(\frac{\Lambda_b-1}{\Lambda_b}\right)^{l-m}\right)\ \mu_l} \tag{55}$$

$$\lim_{N\to\infty} \frac{C_M^{b1c,N}}{C_M^{u,N}} = 2\ \frac{\sum_{l=1}^{\infty}\left(\sum_{m=1}^{l}\ \frac{m}{m+1}\ \binom{l}{m}\left(\frac{1}{\Lambda}\right)^m\left(\frac{\Lambda-1}{\Lambda}\right)^{l-m}\right)\ \mu_l}{\sum_{l=1}^{\infty}\left(\sum_{m=1}^{l}\ \frac{2m-1}{2(m+1)}\ \binom{l}{m}\left(\frac{1}{\Lambda_b}\right)^m\left(\frac{\Lambda_b-1}{\Lambda_b}\right)^{l-m}\right)\ \mu_l} \tag{56}$$

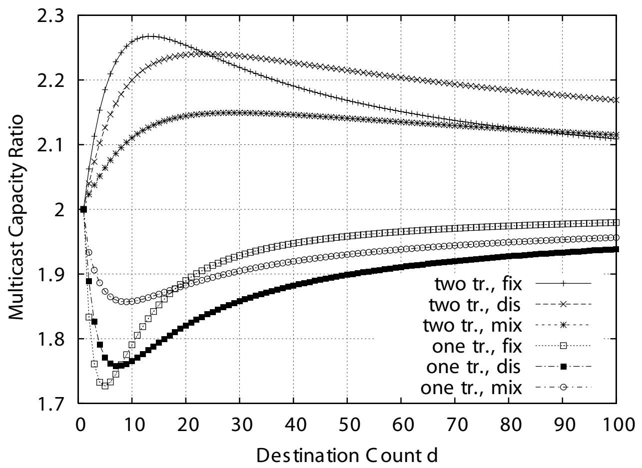

Fig. 3. Multicast capacity ratios $\lim_{N\to\infty} C_M^{b,N}/C_M^{u,N}$ (with two copy transmission in bidir. ring) and $\lim_{N\to\infty} C_M^{b1c,N}/C_M^{u,N}$ (with one copy transmission in bidir. ring) as a function of maximum destination count d; $\Lambda_b = 2$, fixed.

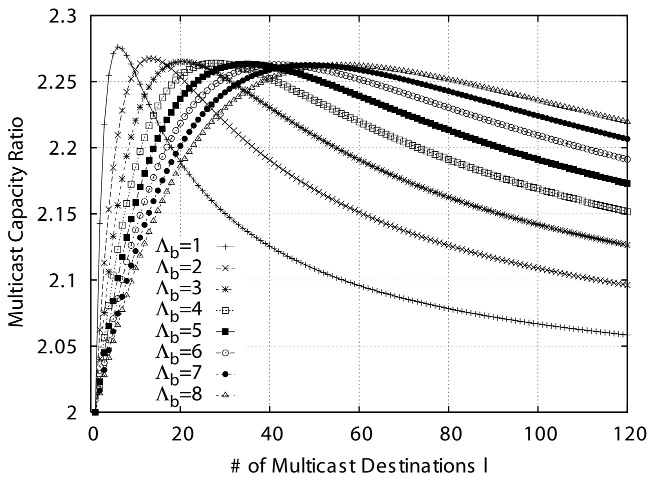

Fig. 4. Multicast capacity ratio $\lim_{N\to\infty} C_M^{b,N}(l)/C_M^{u,N}(l)$ (with two copy transmission in bidirectional ring) for fixed fanout $F = l$ as a function of number of wavelengths Λ_b and destinations l.

We plot these ratios for fixed fanout $\mu_d = 1$ and $\mu_l = 0$ for $l \ne d$ (`fix`), for distributed fanout $\mu_l = 1/d$ for $l = 1, \dots, d$ and $\mu_l = 0$ for $l > d$ (`dis`), and mixed fanout $\mu_1 = 0.75$ and $\mu_l = 0.25/(d-1)$ for $l = 2, \dots, d$, and $\mu_l = 0$ for $l > d$ (`mix`) as a function of the destination count d for networks with a total of $\Lambda = 4$ wavelengths. We observe that the multicast capacity ratios reach the largest (respectively, smallest) values for the fixed fanout. With the distributed fanout, which in a sense averages across all fixed fanouts that are less or equal to d, the ratio is "smoothed" out. Similarly, the mixed fanout combines a unicast ($F = 1$) component with a distributed fanout component; whereby the unicast component (for which the ratio is two) keeps the overall ratio closer to two.

We further plot the multicast capacity ratio $\lim_{N\to\infty} C_M^{b,N}(l)/C_M^{u,N}(l)$ (with two copy transmission in bidir. ring) for fixed fanout $F = l$ as a function of the number of multicast destinations l for different numbers of wavelength channels in each direction Λ_b in Fig. 4.

We observe that with increasing number of wavelength channels Λ_b, the maximum of the plot of $\lim_{N\to\infty} C_M^{b,N}(l)/C_M^{u,N}(l)$ decreases and is reached for an increasing number of multicast destinations. For larger numbers of multicast destinations l (beyond the range of the plot) the multicast capacity ratio drops to two, as expected from Corollary 2.

We proceed to examine the question how large the ratio $\lim_{N\to\infty} C_M^{b,N}/C_M^{u,N}$ gets, i.e., what is the maximum of this ratio for all numbers of wavelengths $\Lambda_b = 1, 2, \dots$ and fanout distributions $\mu_l,\ l \ge 1$? First, we note that it suffices to consider the maximum ratio for all $\Lambda_b = 1, 2, \dots$ and all fixed fanouts $F = l,\ l = 1, 2, \dots$. To see this, observe from the fixed fan out plots in Fig. 4 that for a given Λ_b there is exactly one l_{Λ_b} that attains the maximum ratio, i.e.,

$$\lim_{N\to\infty} \frac{C_M^{b,N}(l)}{C_M^{u,N}(l)} \le \lim_{N\to\infty} \frac{C_M^{b,N}(l_{\Lambda_b})}{C_M^{u,N}(l_{\Lambda_b})} \qquad \forall\, l = 1, 2, \dots \tag{57}$$

$$\lim_{N\to\infty} \frac{C_M^{b,N}}{C_M^{u,N}} = 2\, \frac{\sum_{l=1}^{\infty} \left(\sum_{m=1}^{l} \frac{m}{m+1} \binom{l}{m} \left(\frac{1}{\Lambda}\right)^m \left(\frac{\Lambda-1}{\Lambda}\right)^{l-m} \right) \mu_l}{\sum_{l=1}^{\infty} \left(1 - \sum_{m=0}^{l} \frac{1}{m+1} \left\{ \Psi(m+2) + \gamma \right\} \binom{l}{m} \left(\frac{1}{\Lambda_b}\right)^m \left(\frac{\Lambda_b-1}{\Lambda_b}\right)^{l-m} \right) \mu_l} \quad (58)$$

$$=: 2\, \frac{\sum_{l=1}^{\infty} a_l\ \mu_l}{\sum_{l=1}^{\infty} b_l\ \mu_l} \quad (59)$$

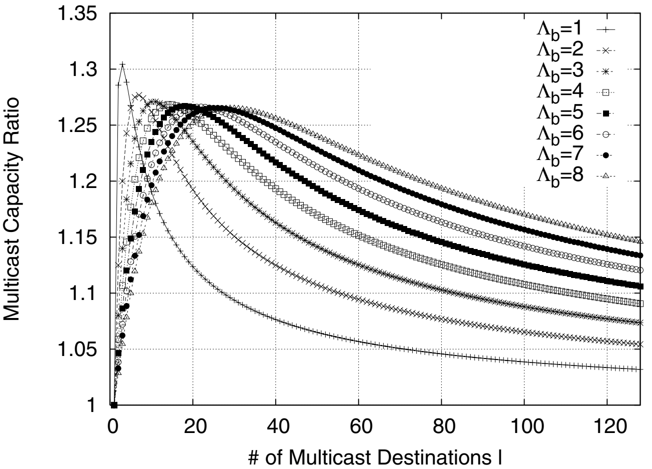

Fig. 5. Capacity ratio (gain) with two copy transmission over one copy transmission in bidiretional ring network $\lim_{N\to\infty} C_M^{b,N}(l)/C_M^{b1c,N}(l)$ for fixed fanout $F = l$ as a function of number of wavelengths Λ_b and destinations l.

Based on this observation we can show that for a fixed Λ_b, the ratio $\lim_{N\to\infty} C_M^{b,N}/C_M^{u,N}$ for any arbitrary distribution μ_l, $l \geq 1$, is at most $\lim_{N\to\infty} C_M^{b,N}(l_{\Lambda_b})/C_M^{u,N}(l_{\Lambda_b})$. In particular, we can rewrite (55) as shown in (58) and (59) (shown on this page). The observation (57) implies in the a_l and b_l notation that

$$\frac{a_l}{b_l} \leq \frac{a_{l_{\Lambda_b}}}{b_{l_{\Lambda_b}}} \qquad \forall\ l = 1, 2, \ldots \quad (60)$$

For an arbitrary fanout distribution μ_l, $l \geq 1$, and the corresponding ratio $\lim_{N\to\infty} C_M^{b,N}/C_M^{u,N}$ we have then (61) through (65) (shown on the next page). Based on our above observation, (60) holds, and we have thus shown that it suffices to consider only fixed fanouts in the maximization. Hence, our maximization problem is expressed in (66) (shown on ther next page). We have not been able to formally solve this maximization problem. However, from numerically evaluating

$$2\, \frac{\sum_{m=1}^{l} \frac{m}{m+1} \binom{l}{m} \left(\frac{1}{\Lambda}\right)^m \left(\frac{\Lambda-1}{\Lambda}\right)^{l-m}}{1 - \sum_{m=0}^{l} \frac{1}{m+1} \left\{ \Psi(m+2) + \gamma \right\} \binom{l}{m} \left(\frac{1}{\Lambda_b}\right)^m \left(\frac{\Lambda_b-1}{\Lambda_b}\right)^{l-m}} \quad (67)$$

as a function of Λ_b and l we obtained the ratios plotted in Fig. 4. We observe that the multicast capacity ratio $\lim_{N\to\infty} C_M^{b,N}(l)/C_M^{u,N}(l)$ reaches $11,235/4,936 \approx 2.27613$ for $\Lambda_b = 1$ and a fanout of $l = 6$ multicast destinations. Analogous evaluations show that the multicast capacity ratio $\lim_{N\to\infty} C_M^{b1c,N}(l)/C_M^{u,N}(l)$ drops to 5/3 for $\Lambda_b = 1$ and a fanout of $l = 2$ multicast destinations.

For bidirectional ring networks, we obtain from the value 2.27613 reached by $\lim_{N\to\infty} C_M^{b,N}(l)/C_M^{u,N}(l)$ with (20) that the relative advantage of two copy transmission is larger than 13.8% in the $\Lambda_b = 1$, $l = 6$ scenario. More specifically, from (47) and (54) we obtain (68) (shown on the next page) and plot the corresponding capacity gain with a fixed fanout $\lim_{N\to\infty} C_M^{b,N}(l)/C_M^{b1c,N}(l)$ in Fig. 5. We find that the gain reaches $30/23 \approx 1.30435$ for $\Lambda_b = 1$ wavelength and a fanout of $l = 3$ multicast destinations.

VI. Conclusion

We have examined the effective multicast capacity (stability limit) of packet-switched ring wavelength division multiplexing (WDM) networks. In particular, we have compared the capacities of bidirectional ring networks (both with two and one copy transmission per wavelength) with the capacity of a unidirectional ring, whereby all networks have the same total number of wavelength channels. With an elementary analysis of the number of traversed critical segments which attain the highest utilization and govern the capacity we have proven fundamental bounds on the ratios of the multicast capacities of the bidirectional ring to the multicast capacity of the unidirectional ring: The bidirectional ring with two copy transmission to unidirectional ring capacity ratio C_M^b/C_M^u is bounded by two from below, whereas the corresponding capacity ratio with one copy transmission in the bidirectional ring C_M^{b1c}/C_M^u is bounded by two from above. The bounds are generally valid for arbitrary mixes of uniform unicast, multicast, and broadcast traffic.

For networks with a large number N of network nodes (formally, $N \to \infty$) we have derived simple closed-form characterizations of the utilization of the critical segments and the effective multicast capacities C_M^u of the unidirectional ring, C_M^b of the bidirectional ring with two copy transmission, and C_M^{b1c} of the bidirectional ring with one copy transmission. Our numerical work demonstrates that these characterizations are in close agreement with the exact capacities for finite N for networks with 20 or more nodes homed on one wavelength. From these characterizations we have formally proven that both capacity ratios C_M^b/C_M^u and C_M^{b1c}/C_M^u approach two for multicasts with a large number of destinations.

We have quantified the capacity gain achieved by two copy transmission over one-copy transmission in bidirectional ring networks and found that this gain is most pronounced for a small number of wavelengths and small to moderate numbers of multicast destinations. These results can form the basis for assessing the trade-offs between increased transmission effort with two-copy transmission and the corresponding capacity gains.

Acknowledgement

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$$\lim_{N\to\infty}\frac{C_M^{b,N}}{C_M^{u,N}} \le \lim_{N\to\infty}\frac{C_M^{b,N}(l_{\Lambda_b})}{C_M^{u,N}(l_{\Lambda_b})} \iff \frac{\sum_{l=1}^{\infty} a_l\ \mu_l}{\sum_{l=1}^{\infty} b_l\ \mu_l} \le \frac{a_{l_{\Lambda_b}}}{b_{l_{\Lambda_b}}} \tag{61}$$

$$\iff b_{l_{\Lambda_b}}\sum_{l=1}^{\infty} a_l\ \mu_l \le\ a_{l_{\Lambda_b}}\sum_{l=1}^{\infty} b_l\ \mu_l \tag{62}$$

$$\iff \sum_{l=1}^{\infty}\left(b_{l_{\Lambda_b}}a_l - a_{l_{\Lambda_b}}b_l\right)\ \mu_l\ \le 0 \tag{63}$$

$$\Longleftarrow \left(b_{l_{\Lambda_b}}a_l - a_{l_{\Lambda_b}}b_l\right)\ \le 0 \qquad \forall l \ge 1 \tag{64}$$

$$\iff \text{Inequality (60) holds} \tag{65}$$

$$\max_{\Lambda_b\ge 1, l\ge 1}\lim_{N\to\infty}\frac{C_M^{b,N}(l)}{C_M^{u,N}(l)} = \max_{\Lambda_b\ge 1, l\ge 1} 2\,\frac{\sum_{m=1}^{l}\ \frac{m}{m+1}\ \binom{l}{m}\left(\frac{1}{\Lambda}\right)^m\left(\frac{\Lambda-1}{\Lambda}\right)^{l-m}}{1-\sum_{m=0}^{l}\ \frac{1}{m+1}\left\{\Psi(m+2)+\gamma\right\}\binom{l}{m}\left(\frac{1}{\Lambda_b}\right)^m\left(\frac{\Lambda_b-1}{\Lambda_b}\right)^{l-m}} \tag{66}$$

$$\lim_{N\to\infty}\frac{C_M^{b,N}}{C_M^{b1c,N}} = \frac{\sum_{l=0}^{\infty}\sum_{m=1}^{l}\ \frac{2m-1}{2(m+1)}\ \binom{l}{m}\left(\frac{1}{\Lambda_b}\right)^m\left(\frac{\Lambda_b-1}{\Lambda_b}\right)^{l-m}\mu_l}{1-\sum_{l=0}^{\infty}\sum_{m=0}^{l}\ \frac{1}{m+1}\left\{\Psi(m+2)+\gamma\right\}\binom{l}{m}\left(\frac{1}{\Lambda_b}\right)^m\left(\frac{\Lambda_b-1}{\Lambda_b}\right)^{l-m}\mu_l} \tag{68}$$

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the drawing of the figures.

Appendix

Appendix A: Proof of Corollaries 2 and 3

For the proof of Corollary 2 we first note that according to (55) we have (69) (shown on the next page). So the claim of Corollary 2 follows from

$$\lim_{l\to\infty}\sum_{m=1}^{l}\ \frac{m}{m+1}\ \binom{l}{m}\left(\frac{1}{\Lambda}\right)^m\left(\frac{\Lambda-1}{\Lambda}\right)^{l-m} \quad = \quad 1, \tag{70}$$

and (71) (shown on the next page).

To prove (70), we write (for the sake of clarity) p_u and p_b in place of $1/\Lambda$ and $1/\Lambda_b$, respectively, and note that

$$\Big|\sum_{m=1}^{l}\ \frac{m}{m+1}\binom{l}{m}p_u^m(1-p_u)^{l-m}-1\Big| \tag{72}$$

$$=\Big|\sum_{m=1}^{l}\ \left(\frac{m}{m+1}-1\right)\binom{l}{m}p_u^m(1-p_u)^{l-m}-(1-p_u)^l\Big| \tag{73}$$

$$\le\sum_{m=1}^{l}\ \frac{1}{m+1}\binom{l}{m}p_u^m(1-p_u)^{l-m}+(1-p_u)^l \tag{74}$$

$$=\frac{p_u^{-1}}{l+1}\ \sum_{m=1}^{l}\ \binom{l+1}{m+1}p_u^{m+1}(1-p_u)^{l+1-(m+1)}+(1-p_u)^l \tag{75}$$

$$\le\frac{p_u^{-1}}{l+1}\ \sum_{m=0}^{l+1}\ \binom{l+1}{m}p_u^m(1-p_u)^{l+1-m}+(1-p_u)^l \tag{76}$$

$$=\frac{p_u^{-1}}{l+1}+(1-p_u)^l. \tag{77}$$

To prove (71), we notice that Ψ increases logarithmically: $\lim_{x\to\infty}\Psi(x)/\log x\ =\ 1$. In particular, $\lim_{x\to\infty}\Psi(x+2)/(x+1)=0$. So we obtain (for l sufficiently large),

$$\sum_{m=1}^{l}\ \frac{1}{m+1}\{\Psi(m+2)+\gamma\}\ \binom{l}{m}p_b^m(1-p_b)^{l-m} \tag{78}$$

$$\le\frac{p_b^{-1}}{l+1}\sum_{m=1}^{l}\ \{\Psi(l+2)+\gamma\}\ \binom{l+1}{m+1}p_b^{m+1}(1-p_b)^{l+1-(m+1)} \tag{79}$$

$$\le\frac{p_b^{-1}}{l+1}\{\Psi(l+2)+\gamma\}\sum_{m=0}^{l+1}\ \binom{l+1}{m}p_b^m(1-p_b)^{l+1-m} \tag{80}$$

$$\le\frac{p_b^{-1}}{l+1}\{\Psi(l+2)+\gamma\}\ \longrightarrow\ 0 \quad \text{as} \quad l\to\infty. \tag{81}$$

For the proof of Corollary 3 we first observe that according to Theorem 4 we have

$$\lim_{N\to\infty}\frac{C_M^{b,N}(l)}{C_M^{u,N}(l)}\ =\ 2\,\frac{\sum_{m=1}^{l}\ \frac{m}{m+1}\ \binom{l}{m}\left(\frac{1}{\Lambda}\right)^m\left(\frac{\Lambda-1}{\Lambda}\right)^{l-m}}{\sum_{m=1}^{l}\ \frac{2m-1}{2(m+1)}\ \binom{l}{m}\left(\frac{1}{\Lambda_b}\right)^m\left(\frac{\Lambda_b-1}{\Lambda_b}\right)^{l-m}}\,. \tag{82}$$

So the claim of Corollary 3 follows from (70) and

$$\lim_{l\to\infty}\sum_{m=1}^{l}\ \frac{2m-1}{2(m+1)}\ \binom{l}{m}\left(\frac{1}{\Lambda_b}\right)^m\left(\frac{\Lambda_b-1}{\Lambda_b}\right)^{l-m} \quad = \quad 1, \tag{83}$$

which is proven similarly to (70).

Appendix B: The case of "escaping" fanout mass

So far we assumed that the approximating fanout distribution $\mu_l^N,\ l\ =\ 1,\ldots,N-1$ has a nondegenerate limit distribution $\mu_l,\ l\ge 1$, that satisfies $\lim_{N\to\infty}\mu_l^N=\mu_l,\ l\ge 1$ as well as $\mu_l\ge 0$ and $\sum_{l=1}^{\infty}\mu_l\ =\ 1$. The question remains how the ratios of the multicast capacities behave if the mass of the approximating fanout distribution escapes to infinity, that is, if

$$\lim_{N\to\infty}\mu_l^N=0 \qquad\qquad \forall\ l\ge 1. \tag{84}$$

$$\lim_{N\to\infty} \frac{C_M^{b,N}(l)}{C_M^{u,N}(l)} = 2 \frac{\sum_{m=1}^{l} \frac{m}{m+1} \binom{l}{m}\left(\frac{1}{\Lambda}\right)^m \left(\frac{\Lambda-1}{\Lambda}\right)^{l-m}}{1-\sum_{m=0}^{l} \frac{1}{m+1}\left\{\Psi(m+2)+\gamma\right\}\binom{l}{m}\left(\frac{1}{\Lambda_b}\right)^m \left(\frac{\Lambda_b-1}{\Lambda_b}\right)^{l-m}} \tag{69}$$

$$\lim_{l\to\infty} \sum_{m=1}^{l} \frac{1}{m+1}\{\Psi(m+2)+\gamma\} \binom{l}{m}\left(\frac{1}{\Lambda_b}\right)^m \left(\frac{\Lambda_b-1}{\Lambda_b}\right)^{l-m} = 0 \tag{71}$$

Informally, (84) means that with increasing N the mass of $\mu_l^N,\ l = 1, \ldots, N-1$ more and more concentrates on large l, that is, small fanout sets are becoming more and more unlikely. One simple illustrative example for an approximating fanout distribution $(\mu_l^N)_{l=1}^{N-1}$ satisfying (84) is:

$$\mu^N_{\lceil p(N-1)\rceil} := 1 \text{ and } \mu_l^N := 0$$
$$\text{for all } l \in \{1, \ldots, N-1\} \setminus \{\lceil p(N-1)\rceil\},$$

where p is some constant in the half-open interval $(0, 1]$. In view of Corollary 3 and Corollary 2, it is straightforward to prove the following theorem.

Theorem 5: Assume (84) holds. Then,

$$\lim_{N\to\infty} \frac{C_M^{b,N}}{C_M^{u,N}} = 2, \text{ and } \lim_{N\to\infty} \frac{C_M^{b1c,N}}{C_M^{u,N}} = 2. \tag{85}$$

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